CIV Entrance Exam 2018 : solutions

Exercise 1

We have $(*) \quad \cos^2 x - \sin x + 1 = 0 \Leftrightarrow 1 - \sin^2 x - \sin x + 1 = 0 \Leftrightarrow \sin^2 x + \sin x - 2 = 0$. Let $X = \sin x$, then $(*) \Leftrightarrow X^2 + X - 2 = 0$. The discriminant is $\Delta = 1^1 - 4(-2) = 9 = 3^2$ so $X = \dfrac{-1 \pm 3}{2} = -2 \text{ or } 1$.

But $X = \sin x \in [-1, 1]$ so $X = 1$.

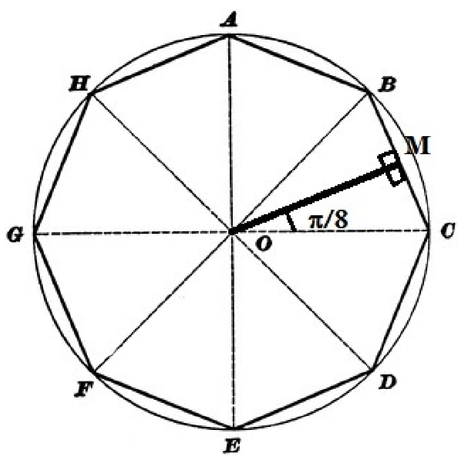
Hence $(*) \Leftrightarrow \sin x = 1 \Leftrightarrow x = \dfrac{\pi}{2}$ because $x \in [0, \pi]$.

We can conclude that $\boxed{\text{there is a unique solution of the equation } (*) \text{ in } [0,\pi] \text{ which is } \frac{\pi}{2}}$.

Exercise 2

We have $\cos^2 \dfrac{\pi}{8} = \dfrac{1}{2}(1 + \cos \dfrac{\pi}{4}) = \dfrac{1}{2}(1 + \dfrac{1}{\sqrt{2}}) = \dfrac{1}{4}(2 + \sqrt{2})$, hence $\left|\cos \dfrac{\pi}{8}\right| = \dfrac{1}{2}\sqrt{2 + \sqrt{2}}$. But $\dfrac{\pi}{8} \in [0, \pi]$, hence $\cos \dfrac{\pi}{8} > 0$, so $\boxed{\cos \frac{\pi}{8} = \frac{1}{2}\sqrt{2 + \sqrt{2}}}$

We have $\sin^2 \dfrac{\pi}{8} = 1 - \cos^2 \dfrac{\pi}{8} = 1 - \dfrac{1}{4}(2 + \sqrt{2}) = \dfrac{1}{4}(2 - \sqrt{2})$, but $\sin \dfrac{\pi}{8} > 0$, hence $\boxed{\sin \frac{\pi}{8} = \frac{1}{2}\sqrt{2 - \sqrt{2}}}$



Let M be the middle of B and C.

We have $AB = BC = 2MC$, but $\sin \dfrac{\pi}{8} = \dfrac{MC}{OC} = MC$,

hence $AB = 2 \sin \dfrac{\pi}{8}$ and $\boxed{AB = \sqrt{2 - \sqrt{2}}}$

Exercise 3

a) $u_4 = (1-i)^3 u_1 = (1-i)^3 = 1 - 3i + 3i^2 - i^3 = 1 - 3i - 3 + i = -2 - 2i$. So $\boxed{u_4 = -2 - 2i}$.

b) We have

$$S_{20} = \sum_{k=1}^{20} u_k = \sum_{k=0}^{19} (1-i)^k = \sum_{k=0}^{19} \left(\sqrt{2} e^{-i\frac{\pi}{4}}\right)^k = \frac{(\sqrt{2})^{20} e^{-i\frac{\pi}{4} \times 20} - 1}{\sqrt{2} e^{-i\frac{\pi}{4}} - 1} = \frac{2^{10} e^{-5i\pi} - 1}{1 - i - 1} = i(2^{10}(-1) - 1)$$

hence $\boxed{S_{20} = -i(1 + 2^{10})}$

c) For all $n \geqslant 1$, we have $v_{n+1} = u_{n+1}u_{n+1+k} = (1-i)u_n(1-i)u_{n+k} = (1-i)^2 v_n = (1-2i+i^2)v_n$. Hence $v_{n+1} = -2iv_n$. So $\boxed{(v_n) \text{ is a geometric sequence with reason } -2i}$.

d) (i) For all $n \geqslant 1$, we have $w_{n+1} = |u_{n+1} - u_{n+2}| = |(1-i)u_n - (1-i)u_{n+1}|$
$= |(1-i)(u_n - u_{n+1})| = |1-i| \times |u_n - u_{n+1}| = \sqrt{2}w_n$.
Hence $\boxed{(w_n) \text{ is a geometric sequence with reason } \sqrt{2}}$.

(ii) The sequence (u_n) references to points on a spiral, each point obtained from the previous one by a rotation of $\frac{\pi}{4}$, and multiplying its distance to origin by $\sqrt{2}$. The sequence (w_n) is the sequence of distances between a point and the next point in the previous sequence : it follows from (i) that this distance is multiplied by $\sqrt{2}$ at each step.

Exercise 4

Take for example

$$f(x) = \frac{1}{1 + \frac{x^2}{12}}$$

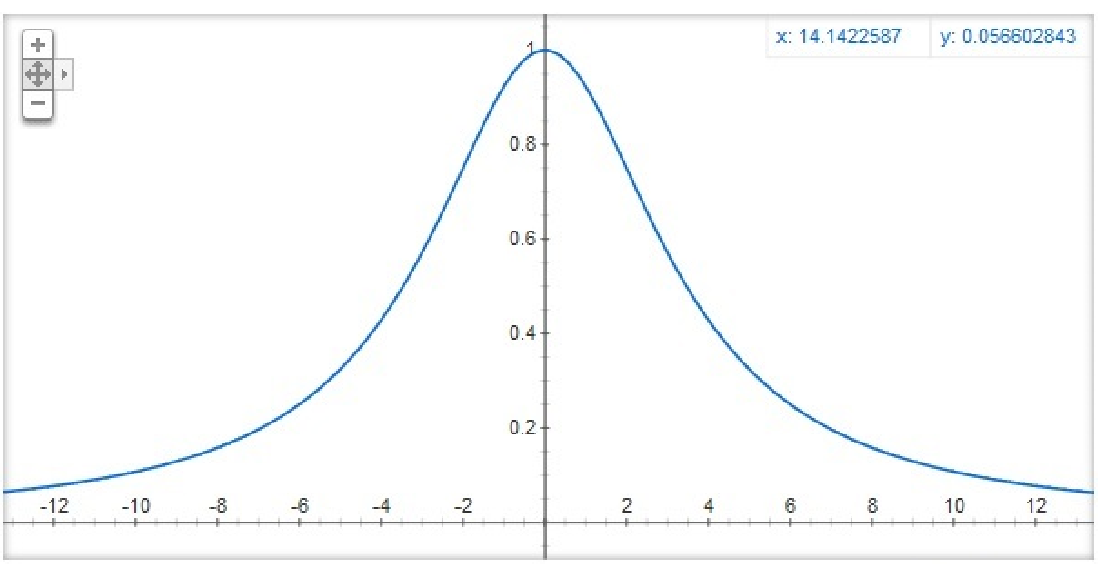
$f(x)$ is defined for all $x \in \mathbb{R}$.
As quotient and sum of functions that can be differentiated infinitely many, f can be differentiated infinitely many.
Clearly, $f(x) > 0$ for all $x \in \mathbb{R}$.
We have

$$f'(x) = -\frac{24x}{(x^2+12)^2}$$

and

$$f''(x) = \frac{72(x^2-4)}{(x^2+12)^3}$$

so $f'(x) = 0 \Leftrightarrow x = 0$, hence the tangent at $x = 0$ is horizontal,
and $f''(x) = 0 \Leftrightarrow x^2 = 4 \Leftrightarrow x = \pm 2$, which is the case when the points at $x = 2$ and $x = -2$ are inflexion points, i.e. the tangent crosses the curve of f : at these points, the curvature of f changes its direction.



Exercise 5

The idea is to write a number $0 \leqslant n \leqslant 35$ in basis 6 : then $n = c_1 6^0 + c_2 6^1$ with digits from 0 to 5. This decomposition is then unique. But here we have n_1 and n_2 from 1 to 6, and we want n from 1 to 36, so take $c_1 = n_1 - 1, c_2 = n_2 - 1$, and then we get

$$n = 1 + (n_1 - 1) + 6(n_2 - 1) = -6 + n_1 + 6n_2$$

So $\boxed{a = -6, b = 1, c = 6}$.

The law is uniform because each number from 1 to 36 corresponds to a unique couple (n_1, n_2), due to the unicity of decompostion in basis 6, and the fact that each dice is fair.

Exercise 6

We have

$$\begin{array}{llcl}
\multicolumn{4}{l}{\displaystyle\sum_{k=0}^{n-1} ((k+1)(k+2)(k+3)(k+4) - k(k+1)(k+2)(k+3)) =} \\
= & \ \ 1 \times 2 \times 3 \times 4 & - & 0 \times 1 \times 2 \times 3 \\
 & +2 \times 3 \times 4 \times 5 & - & 1 \times 2 \times 3 \times 4 \\
 & +3 \times 4 \times 5 \times 6 & - & 2 \times 3 \times 4 \times 5 \\
 & \dots & & \\
 & +(n-1)n(n+1)(n+2) & - & (n-2)(n-1)n(n+1) \\
 & +n(n+1)(n+2)(n+3) & - & (n-1)n(n+1)(n+2) \\
\hline \\
= & n(n+1)(n+2)(n+3) & - & 0 \times 1 \times 2 \times 3 \\
= & n(n+1)(n+2)(n+3) & &
\end{array}$$

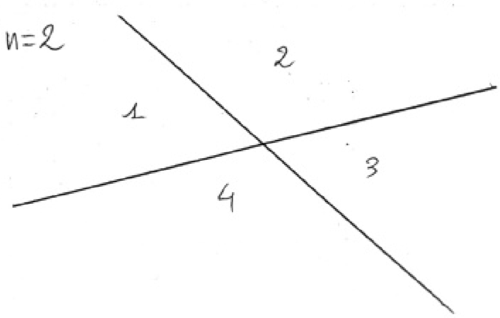
but also

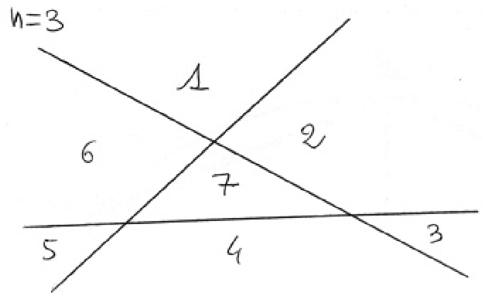
$$\begin{aligned}
&\sum_{k=0}^{n-1} ((k+1)(k+2)(k+3)(k+4) - k(k+1)(k+2)(k+3)) \\
&= \sum_{k=0}^{n-1} (k+1)(k+2)(k+3)(k+4-k) \\
&= \sum_{k=0}^{n-1} 4(k+1)(k+2)(k+3) \\
&= \sum_{k=1}^{n} 4k(k+1)(k+2) \\
&= 4S_n
\end{aligned}$$

Hence $\boxed{S_n = \dfrac{n(n+1)(n+2)(n+3)}{4}}$.

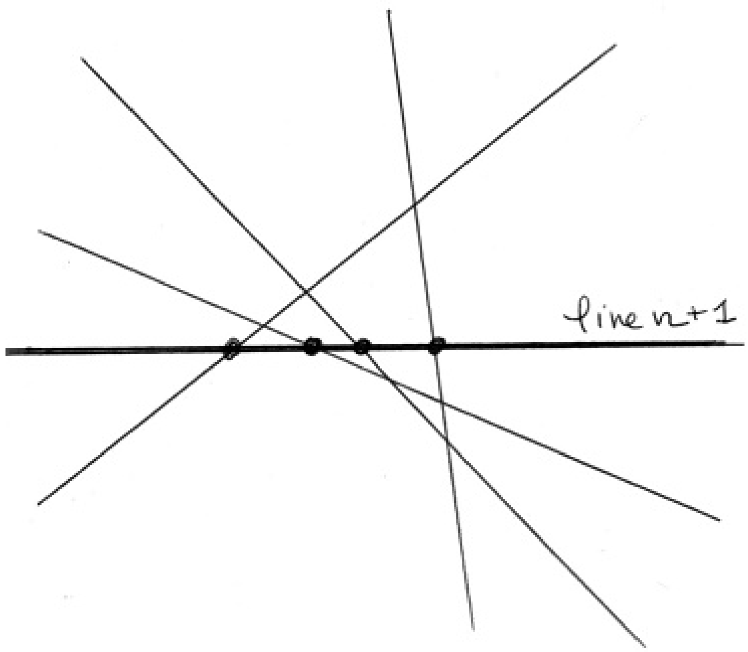
Exercise 7

a) with a drawing, we get $\boxed{r_2 = 4, r_3 = 7}$:





b) the line $n+1$ intersects the other n lines in n points :



which separate this line into $n+1$ intervals (segments or half lines) : each interval cuts a region into two new regions : there is then $n+1$ more regions, so $\boxed{r_{n+1} = r_n + n + 1}$.

c) we prove by induction $\boxed{P(n) :\ r_n = 1 + \sum_{k=0}^{n} k = 1 + \frac{n(n+1)}{2}}$:

for $n = 0$: $r_0 = 1 = 1 + 0$, and $P(0)$ is true.
suppose $P(n)$ is true for n, let us prove $P(n+1)$:
we have $r_{n+1} = r_n + n + 1 = 1 + \sum_{k=0}^{n} k + n + 1$ by induction hypothesis $P(n)$,

hence $r_{n+1} = 1 + \sum_{k=0}^{n+1} k$, which is $P(n+1)$. Which completes the proof by induction.

Exercise 8

There is 2^{10} subfamilies of the family $x_1, x_2, ..., x_{10}$, then there is $2^{10} - 1 = 1023$ non-empty subfamilies. The sum of a subfamily is at most 10×100, because $x_1, ..., x_n \leqslant 100$, and is positive, so there is at most 1000 possible sums for the 1023 non-empty subfamilies.
The pigeonhole principle (or Dirichlet's principle) tells us that two different subfamily, say A and B,

must have the same sum. Remove the common numbers from these families, and we obtain two non empty disjoined subfamilies A' and B' with the same sum QED.

Exercise 9