CIV Entrance Exam 2018
Mathematics

Length: 3 hours

Examination Center: ____________________

Candidate's registration number : ☐☐☐☐☐☐☐☐☐☐

Instructions to candidates

- Write above your registration number and the name of the city where you are taking this exam (Astana, Almaty or Chymkent).
- Do not open the examination paper until instructed to do so.
- No calculators, tables or formula sheets may be used.
- Answers should be written in **English or French** on the question sheet in the space provided.
- A partial answer is always interesting. Don't hesitate to write down your ideas, even incomplete.
- Marks are indicated at the start of each exercice. You are advised to divide your time according to the marks allocated.
- The maximum mark for this examination paper is 125.

Notations used in this document are the usual ones : for example $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ the set of integers, $\mathbb{R}$ the set of real numbers, $\mathbb{C}$ the set of complex numbers, i is the imaginary number such that $i^2 = -1$ and $\frac{df}{dx}$ or $f'(x)$ is the derivative of a function f with respect to x.

BONNE CHANCE / GOOD LUCK !

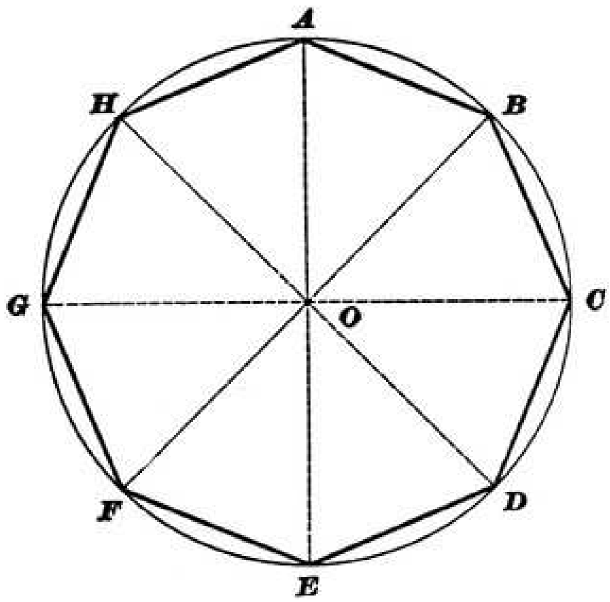
1. [Maximum mark: 5]

Solve $\cos^2 x - \sin x + 1 = 0$ for x in the range $0 \leq x \leq \pi$.

2. [Maximum mark: 8]

Using $\cos 2x = 2(\cos x)^2 - 1$, express the exact values of $\cos \frac{\pi}{8}$ and $\sin \frac{\pi}{8}$ with square roots.

Then use $\sin \frac{\pi}{8}$ to express the length AB of the edges of a regular octogon whose vertices lie on a circle with radius $OA = 1$ and centered at origin O.



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3. [Maximum mark: 15]

A geometric sequence $(u_n)_{n\geq 1}$, with complex terms, is defined by $u_1 = 1$ and $u_{n+1} = (1-i)u_n$.

(a) Find the fourth term of the sequence, giving your answer in the form $x + iy$, $x, y \in \mathbb{R}$.

(b) Find the sum of the first 20 terms of (u_n), giving your answer in the form $a(1+2^m)$, where $a \in \mathbb{C}$, and $m \in \mathbb{Z}$ are to be determined.

(c) Given $k \in \mathbb{N}$, a second sequence $(v_n)_{n\geq 1}$ is defined by $v_n = u_n u_{n+k}$.
Show that (v_n) is a geometric sequence.

(d) A third sequence $(w_n)_{n\geq 1}$ is defined by $w_n = |u_n - u_{n+1}|$.

(i) Show that (w_n) is a geometric sequence.

(ii) State the geometric significance of this result with reference to points on the complex plane.

4. [Maximum mark: 5]

Sketch[1] an example of a real function f defined for all real arguments x. It should have all following properties :

1. f can be differentiated twice,
2. $f(x) > 0$ for all x,
3. $\frac{df}{dx} = 0$ only for $x = 0$,
4. $\frac{d^2 f}{dx^2} = 0$ only for $x = -2$ and $x = 2$.

[1] to sketch = make a rough drawing of.

5. [Maximum mark: 15]

The numbers shown by two six-sided fair dice are labelled n_1 and n_2.
We thus consider the expression : $S = a + b\ n_1 + c\ n_2$ where $(a, b, c) \in \mathbb{R}^3$.
Explaining in detail your proof, find the values of a, b, c such as the range of possible values for S covers all integers from 1 to 36 following a uniform law.

6. [Maximum mark: 15]

For $n \in \mathbb{N}^*$, evaluate $S_n = 1 \times 2 \times 3 \quad + \quad 2 \times 3 \times 4 \quad + \quad \ldots \quad + \quad n \times (n+1) \times (n+2)$.
(for example $S_1 = 6$, $S_2 = 30$, $S_3 = 90$).

Explain your calculations in detail.

7. [Maximum mark: 15]

Let us consider n lines in the plane $\mathbb{R}^2$ such that any two are not parallel, and any three do not intersect. They separate the plane into r_n contiguous regions. For example, one has $r_0 = 1$, $r_1 = 2$.

a) Give the values of r_2, r_3.

b) Express r_{n+1} in terms of r_n.

c) Give r_n as a fonction of n, and prove this expression by induction.

8. [Maximum mark: 25]

Given 10 natural numbers $x_1, ..., x_{10}$ greater or equal to 1 and lesser or equal to 100, show that there always exists two disjointed non-empty subfamilies A and B with the same sum.

For example, with $(x_1, ..., x_{10}) = (1, 2, 3, 4, 5, 10, 11, 12, 13, 100)$, one can take $A = (4, 10)$ and $B = (1, 2, 11)$, because $4 + 10 = 1 + 2 + 11$.

Be aware that the numbers $x_1, ..., x_{10}$ are not necessarily distinct.

9. [Maximum mark: 25]

Given any 1000 distinct points in the plane $\mathbb{R}^2$, demonstrate that a line separating the plane into two regions containing 500 points on each side always exists.