# CIV Entrance Exam 2019
# Mathematics

Length: 3 hours

Candidate's registration number : ☐☐☐☐☐☐☐☐☐☐

**Instructions to candidates**

- Write above your registration number.
- Do not open the examination paper until instructed to do so.
- No calculators, tables or formula sheets may be used.
- Answers should be written in **english or french** on separate provided answer sheets.
- Do not forget to fill out your registration number and the correct page numbering (page .. out of ..) on each of your answer sheets (for example : page 1/5, 2/5 ... 5/5).
- A partial answer is always interesting. Don't hesitate to write down your ideas, even incomplete.
- Marks are indicated at the start of each exercice. You are advised to divide your time according to the marks allocated.
- The maximum mark for this examination paper is 100.

Notations used in this document are the usual ones: for example $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ the set of integers, $\mathbb{R}$ the set of real numbers, $\mathbb{C}$ the set of complex numbers, $i$ is the imaginary number such that $i^2 = -1$ and $\frac{df}{dx}$ or $f'(x)$ is the derivative of a function $f$ with respect to $x$.

BONNE CHANCE / GOOD LUCK !

## 1. [Maximum mark: 5]

Given that $\cos(2\theta) = \frac{1}{9}$, find $\cos\theta$ in the form $\pm\frac{\sqrt{a}}{b}$, where $a, b \in \mathbb{N}$.

## 2. [Maximum mark: 10]

1. Find $z \in \mathbb{C}$ such that: $|z + 3i| = 3|z|$
2. Find $z \in \mathbb{C}$ such that: $\overline{z} = i(z - 1)$

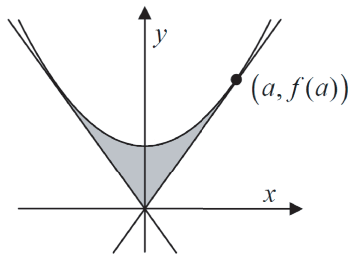
## 3. [Maximum mark: 7]

1. Show that $\frac{1}{\sqrt{r+1}-\sqrt{r}} = \sqrt{r+1} + \sqrt{r}$, for $r \geq 0$.
2. Evaluate $\displaystyle\sum_{r=1}^{99} \frac{1}{\sqrt{r+1}+\sqrt{r}}$.

## 4. [Maximum mark: 8]

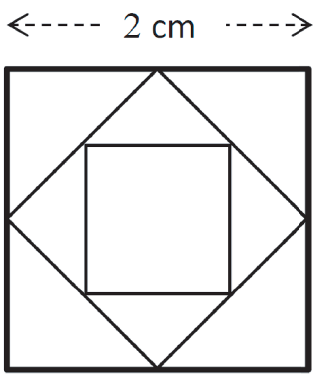
Let $c$ be a positive constant. For $x \in \mathbb{R}$, the function $f$ is given by $f(x) = x^2 + c$.

1. The tangent to the curve $y = f(x)$ at the point $(a, f(a))$ passes through the origin, where $a > 0$. Express $a$ in terms of $c$.
2. The tangent at $(-a, f(-a))$ also passes through the origin. Find, in terms of $c$, the area of the region enclosed by these two tangents and the curve.




## 5. [Maximum mark: 15]

The side of a square is 2 cm. Joining the midpoints of all sides makes an inner square and this process goes to infinity.
The following picture shows the drawing of the first three squares.




What is the sum of the perimeters of all the squares?
Give your answer in the form $a + b\sqrt{c}$ cm, where $a, b$ and $c \in \mathbb{N}$.

## 6. [Maximum mark: 12]

Luke is a web customer of CivB (the Civ Bank). In order to access to his bank account online, Luke has been given by CivB a unique six-digits password.
A password may contain any digits from 0 through 9 and digits may be repeated. For example, the following is a valid password

$$\boxed{0}\quad\boxed{6}\quad\boxed{2}\quad\boxed{3}\quad\boxed{6}\quad\boxed{6}$$

1. How many different passwords are possible?
2. (a) How many different passwords do **not** contain **any** zero?
   (b) Assuming that passwords are uniformly distributed, what is the probability that a password contains at least one zero?
3. In fact, each time Luke wants to access his account online, the CivB's website requires him to enter three digits of his password into randomly selected boxes. For example, he may be asked for the $1^{st}$, $4^{th}$ and $5^{th}$ digits, as shown below.

   $$\boxed{\phantom{*}}\quad\boxed{*}\quad\boxed{*}\quad\boxed{\phantom{*}}\quad\boxed{\phantom{*}}\quad\boxed{*}$$

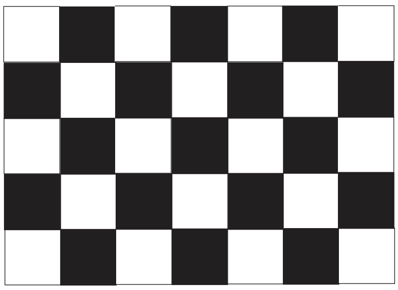
   In how many different ways can CivB select the three required boxes?

## 7. [Maximum mark: 23]

1. Find the minimum value of $x^x$ for $x$ a positive real number.
2. If $x$ and $y$ are positive real numbers, show that $x^y + y^x > 1$.

## 8. [Maximum mark: 20]

Can a $5 \times 7$ checkerboard be covered by L's (that is figures formed from a $2 \times 2$ square by removing one of its four $1 \times 1$ corners) in several layers so that each square of the board is covered by the same number of L's (and of course without going out of the checkerboard)?



BONNE CHANCE / GOOD LUCK !